MATH 5061 Lecture on 3/4/2020

[Annoucement : PS 2 due today, PS 3 up.]

<u>Recall</u>: $E^r \xrightarrow{\pi} M^m$ D : connection on E

$D : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$

$D_X s$ covariant derivative of $s \in \Gamma(E)$ along $X \in \mathfrak{X}(M)$.

- tensorial in X, Leibniz in S

<u>locally</u>: • fix local "frame" $(s_1, \ldots, s_r) = \underline{\underline{s}}$

$$D \underline{\underline{s}} = \underline{\underline{s}}\, \omega \qquad \text{i.e.} \qquad D_{X} s_\alpha = \sum_\beta \omega_\alpha^\beta(X)\, s_\beta$$

$$D \overset{\text{locally}}{=\!=} (\omega_\alpha^\beta) \quad \text{connection matrix of 1-forms} \qquad \omega = \sum a_i\, dx^i$$

(or 1-forms matrix-valued)

- take local frame $e_1, \ldots, e_m$ of $\mathfrak{X}(M)$

$$\rightsquigarrow \quad \omega_\alpha^\beta = \sum_{i=1}^m \Gamma_{\alpha i}^\beta e_i^* \qquad \text{where} \quad \Gamma_{\alpha i}^\beta := \omega_\alpha^\beta(e_i)$$

"Christoffel symbols".

Q: How to define "curvature" of D ?

A: "curvature" = "non-commutativity of covariant derivatives".

<u>Def$^{\underline{n}}$</u> : Given a connection D on a vector bundle $\pi : E \to M$, we define the **curvature of D** as a map :

For any $X, Y \in \mathfrak{X}(M)$, we take

$$R(X,Y) : \Gamma(E) \longrightarrow \Gamma(E) \qquad \text{"curvature tensor"}$$

defined by
$$\boxed{R(X,Y)(s) := \underbrace{D_X(D_Y s) - D_Y(D_X s)}_{\text{commutativity of cov. derivatives}} - D_{[X,Y]} s}$$

to ensure that it is a "tensor".

Note: Clearly $R(X,Y) = -R(Y,X)$, so $R \in \Gamma(\Omega^2(M) \otimes \mathrm{End}(E))$
X, Y

Need to check: $\begin{cases} R(fX,Y) \overset{?}{=} f R(X,Y) \\ R(X,Y)(fs) \overset{\text{Ex:}}{=} f R(X,Y) \end{cases}$ $\forall f \in C^\infty(M)$

Pf:
$$\begin{aligned} R(fX,Y)(s) &= D_{fX}(D_Y s) - D_Y(D_{fX} s) - D_{[fX,Y]} s \\ &= f D_X(D_Y s) - D_Y(f D_X s) - D_{f[X,Y] - Y(f)X}\, s \\ &= f D_X D_Y s - f D_Y D_X s - \cancel{Y(f) D_X s} \\ &\quad - f D_{[X,Y]} s + \cancel{Y(f) D_X s} \\ &= f R(X,Y)(s) \end{aligned}$$
cancel

So, we have established the following:

FACT: The curvature tensor R is $\mathrm{End}(E)$-valued 2-forms.

Locally: R = curvature matrix of 2-forms

$\underline{s} = (s_1, \dots, s_r)$ local frame of E

$\Omega = (\Omega_\alpha^\beta)$ $(r \times r)$-matrix of 2-forms.

$$R(X,Y)(s_\alpha) = \sum_\beta \Omega_\alpha^\beta(X,Y)\, s_\beta$$

Lemma: $\boxed{\Omega = d\omega + \omega \wedge \omega}$ (*) where $\omega = (\omega_\alpha^\beta)$ matrix of 1-forms

i.e. $\Omega_\alpha^\beta = d\omega_\alpha^\beta + \sum_\gamma \omega_\gamma^\beta \wedge \omega_\alpha^\gamma$

Proof: Recall: $D s_\alpha = \omega_\alpha^\beta s_\beta$.

$$\begin{aligned} D_X(D_Y s_\alpha) &= D_X(\omega_\alpha^\beta(Y) s_\beta) \\ &= X(\omega_\alpha^\beta(Y))\, s_\beta + \omega_\alpha^\beta(Y) D_X s_\beta \\ &= X(\omega_\alpha^\beta(Y))\, s_\beta + \omega_\alpha^\gamma(Y)\, \omega_\gamma^\beta(X)\, s_\beta \\ &= [X(\omega_\alpha^\beta(Y)) + \omega_\alpha^\gamma(Y)\, \omega_\gamma^\beta(X)]\, s_\beta \end{aligned}$$

$$- D_Y(D_X s_\alpha) = [Y(\omega_\alpha^\beta(X)) + \omega_\alpha^\gamma(X)\, \omega_\alpha^\beta(Y)]\, s_\beta$$

$$- D_{[X,Y]} s_\alpha = \omega_\alpha^\beta([X,Y])\, s_\beta$$

$$R(X,Y)(s_\alpha) = \left[X(\omega_\alpha^\beta(Y)) - Y(\omega_\alpha^\beta(X)) - \omega_\alpha^\beta([X,Y]) \right] s_\beta$$

$$+ \left[\omega_\gamma^\beta(X)\omega_\alpha^\gamma(Y) - \omega_\gamma^\beta(Y)\,\omega_\alpha^\gamma(X) \right] s_\beta$$

$$= \underbrace{\left[d\omega_\alpha^\beta(X,Y) + \sum_\gamma \omega_\gamma^\beta \wedge \omega_\alpha^\gamma(X,Y) \right]}_{\Omega_\alpha^\beta(X,Y)} s_\beta .$$

$\square$

Alternative proof that Ω is a tensor using (*)

local frame of E : $\underline{s} = (s_1, \dots, s_r)$ another local frame of E : $\underline{\tilde{s}} = (\tilde{s}_1, \dots, \tilde{s}_r)$

They are related as : $\underline{\tilde{s}} = \underline{s} A$, ie. $\tilde{s}_\alpha = \sum_\beta a_\alpha^\beta s_\beta$

where $A = (a_\alpha^\beta)$ change of basis matrix (of functions)

Notation:

	D	R
$\underline{s}$	$\omega = (\omega_\alpha^\beta)$	$\Omega = (\Omega_\alpha^\beta)$
$\underline{\tilde{s}}$	$\tilde{\omega} = (\tilde{\omega}_\alpha^\beta)$	$\tilde{\Omega} = (\tilde{\Omega}_\alpha^\beta)$

Prop: (Transformation law for ω & Ω)

(i) $\tilde{\omega} = A^{-1}\omega A + A^{-1} dA$ (Note: $\Rightarrow$ ω is NOT a tensor!)

(ii) $\tilde{\Omega} = A^{-1}\Omega A$ (Note: $\Rightarrow$ Ω is a tensor!)

Proof: Recall: $D\underline{s} = \underline{s}\,\omega$ and $D\underline{\tilde{s}} = \underline{\tilde{s}}\,\tilde{\omega}$. But: $\underline{\tilde{s}} = \underline{s} A$

$$\underline{\tilde{s}}\,\tilde{\omega} = D\underline{\tilde{s}} = D(\underline{s} A)$$

$$= (D\underline{s}) A + \underline{s}\, dA$$

$$= (\underline{s}\,\omega) A + \underline{s}\, dA$$

$$= \underline{\tilde{s}} \left[A^{-1}\omega A + A^{-1} dA \right]$$

This proves (i).

Recall: $\Omega = d\omega + \omega\wedge\omega$ and $\tilde{\Omega} = d\tilde{\omega} + \tilde{\omega}\wedge\tilde{\omega}$

$$\tilde{\Omega} = d\left(A^{-1}\omega A + A^{-1}dA\right) + \left(A^{-1}\omega A + A^{-1}dA\right)\wedge\left(A^{-1}\omega A + A^{-1}dA\right)$$

Fact: $d(A^{-1}) = -A^{-1}dA\,A^{-1}$

$$= (-A^{-1}dA\,A^{-1})\wedge(\omega A + dA)$$
$$+ A^{-1}(d\omega\,A - \omega\wedge dA)$$
$$+ A^{-1}(\omega\wedge\omega)A + A^{-1}\omega\wedge dA + A^{-1}dA\,A^{-1}\wedge\omega A + A^{-1}dA\,A^{-1}\wedge dA$$

$$= A^{-1}(d\omega + \omega\wedge\omega)A = A^{-1}\Omega A.$$

$\square$

1st Bianchi Identity: $\boxed{d\Omega = \Omega\wedge\omega - \omega\wedge\Omega}$ $(\#)$

Proof: Locally, $\Omega = d\omega + \omega\wedge\omega$.

$$d\Omega = \cancel{d^2\omega}^{\,0} + d\omega\wedge\omega - \omega\wedge d\omega$$
$$= (\Omega - \omega\wedge\omega)\wedge\omega - \omega\wedge(\Omega - \omega\wedge\omega)$$
$$= \Omega\wedge\omega - \omega\wedge\Omega$$

$\square$

Remark: $(\#)$ $\Leftrightarrow$ "$D\Omega = 0$" for the some "induced connection" D on $\Omega^2(M)\otimes \underbrace{\mathrm{End}(E)}_{E\otimes E^*}$ (HW3)

E.g.) $E \to M$, D: connection $\rightsquigarrow$ D: connection on $E^r_s = \overbrace{E\otimes\cdots\otimes E}^{r}\otimes\underbrace{E^*\otimes\cdots\otimes E^*}_{s} \to M$

s.t. (i) D is a derivation w.r.t. $\otimes$

(ii) $D\circ C = C\circ D$

where C = contraction

Ex: $\theta\in\Gamma(E^*)$, $(D_X\theta)(s) = X(\theta(s)) - \theta(D_X s)$.

Note: $\omega = (\omega_\alpha^\beta)$ has no "geometric meaning" pointwise.

Recall: We can always make $\Gamma_{ij}^k(p) = 0$ by choosing suitable coord.

Lemma: (Existence of "normal coordinates" at $p \in M$)

For any connection D on a vector bundle E over M, fix $p \in M$,

$\Rightarrow \exists$ local frame $\underline{s} = (s_1, \dots, s_r)$ near p s.t.

$$D s_\alpha(p) = 0 \quad \text{for } \alpha = 1, \dots, r.$$

i.e. $\omega_\alpha^\beta(p) = 0$ (or $\Leftrightarrow$ $\Gamma_{\alpha i}^\beta(p) = 0$)

Sketch of Proof: Fix ANY frame $\underline{\tilde{s}} \rightsquigarrow \tilde{\omega} = (\tilde{\omega}_\alpha^\beta)$ connection w.r.t. $\underline{\tilde{s}}$

Goal: Find another local frame $\underline{s} = \underline{\tilde{s}} A$ ← unknown s.t. $\omega = (\omega_\alpha^\beta)$ conn. w.r.t $\underline{s}$

$$\omega = A^{-1} \tilde{\omega} A + A^{-1} dA$$
$$= A^{-1} (\tilde{\omega} A + dA)$$

Choose A s.t. $(\tilde{\omega} A + dA)(p) = 0$. □

Q: When does a connection D exist on a vector bundle?

A: Yes, actually in abundance.

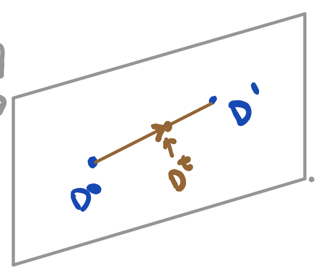
Why? • connection exists locally (as $\omega = (\omega_\alpha^\beta)$)

• can be pieced together to get a globally-defined connection using (1) partition of unity

(2) the space of connection is "convex".

About (2): D^0, D^1 connections on the same bundle E $\Rightarrow$ $D^t = (1-t) D^0 + t D^1$ is also a connection on E

$\mathcal{C} := \{\text{connection } D \text{ on } E\}$ is an affine space.



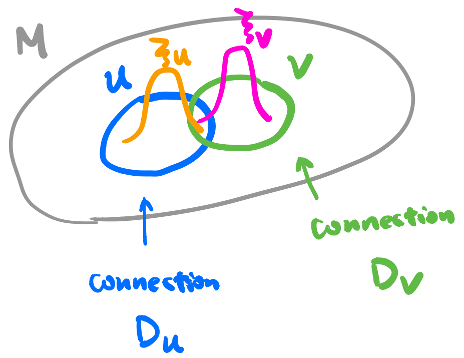
Recall:

$$\tilde{\omega} = A^{-1} \omega A + A^{-1} dA$$
$$\tilde{\omega} - \tilde{\eta} = A^{-1} (\omega - \eta) A$$

$$\mathcal{C} = \{ D = D^0 + \theta \mid \theta \in \underbrace{\Gamma(\Omega^1(M) \otimes \mathrm{End}(E))}_{\text{a vector space } / \mathbb{R}} \}$$

Thm: Given $\pi: E \to M$, there exists a connection D on E.

"Sketch of Proof":



$M = \bigcup_\alpha U_\alpha$ locally finite covering.

$\{\xi_\alpha\}_\alpha$ partition of 1 subordinate to $\{U_\alpha\}$.

ie $\operatorname{spt} \xi_\alpha \subseteq U_\alpha$ and $\sum_\alpha \xi_\alpha(x) = 1 \ \forall x \in M$

$\Rightarrow$ $D := \sum_\alpha \xi_\alpha D_{U_\alpha}$ defines a global connection

$\blacksquare$

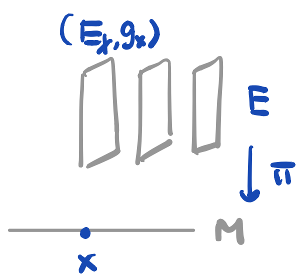
Fiber metrics on vector bundles

Setup:

$$\begin{array}{ccc} \mathbb{R}^r & \to & E \\ & & \downarrow \pi \\ & x \in & M \end{array}$$

g : metric on E

if $\forall x \in M$, g_x defines an "inner product" on E_x



Def^n: We say that a connection D is compatible with g if $\boxed{Dg \equiv 0}$ $(\dagger)$

i.e. $g \in \Gamma(E^* \otimes E^*)$ is parallel

$(\dagger) \iff X(g(s_1, s_2)) = g(D_X s_1, s_2) + g(s_1, D_X s_2)$

$\forall X \in \mathfrak{X}(M), \ \forall s_1, s_2 \in \Gamma(E)$

Q1: $\exists$ fiber metric g on E? **A1**: Yes, in abundance.

Q2: Given g on E, $\exists$ connection D compatible with g? **A2**: later.

Prop: Let D is a connection on E compatible with a fiber metric g on E.

Then, $\boxed{\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}}$ where $\Omega_{\alpha\beta} = g_{\alpha\gamma} \Omega^\gamma_\beta$

2-forms with value in $E^* \otimes E^*$

Here, $g_{\alpha\beta} := g(s_\alpha, s_\beta)$

Remark: If $\underline{\underline{s}} = (s_1, \dots, s_r)$ is local orthonormal (w.r.t g) frame of E,

then $\omega^t = -\omega$ and $\boxed{\Omega^t = -\Omega}$.

"Proof": "Differentiate twice": $dg_{\alpha\beta} = \langle Ds_\alpha, s_\beta \rangle + \langle s_\alpha, Ds_\beta \rangle$ (Ex.) $\square$

Connections on frame bundles

Idea: $\mathbb{R}^r$ r-dim vector space

$$GL(r) = \{ \text{basis on } \mathbb{R}^r \} = \{ A \in M_{r\times r}(\mathbb{R}) : A \text{ invertible} \}$$

why?

Fix a basis $\underline{\underline{s}} \longrightarrow$ all other basis $\tilde{\underline{\underline{s}}} = \underline{\underline{s}} A$

Consider the "frame bundle" of E:

$$\begin{array}{ccc} GL(r) \longrightarrow & F(E) \\ & \downarrow \pi \\ & p \in M \end{array}$$

$F(E)_p := \{ (p, s_1, \dots, s_r) \mid s_1, \dots, s_r \text{ basis of } E_p \}$

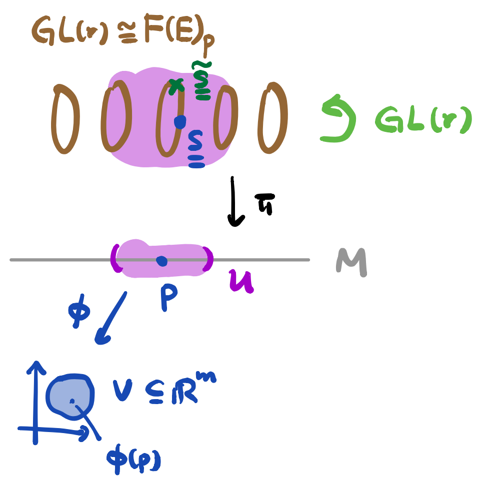
is principal $GL(r)$-bundle over M.

s.t. (1) $\exists$ local charts $\hat{\phi} : \pi^{-1}(U) \to V \times GL(r)$

$$\hat{\phi}(p, \tilde{\underline{\underline{s}}}) = (\phi(p), A)$$

where $\tilde{\underline{\underline{s}}} = \underline{\underline{s}} A$

fixed local frame.



(2) $\exists$ right action of $GL(r)$ on $F(E)$

fiberwise: $\forall C \in GL(r)$,

$$\exists\, R_C : F(E)_p \to F(E)_p.$$

[Reference: Kobayashi–Nomizu "Foundations of DG I"]

Goal: Understand connections from the view point of frame bundle

(or more general, on principal G-bundle)

Recall:

$$\begin{array}{c} E^r \\ \downarrow \\ \underset{\text{open}}{\mathcal{U}} \subseteq M \end{array} \qquad D: \text{connection}$$

$\underline{s} = s_1, \dots, s_r$ local frame over $\mathcal{U}$

$\rightsquigarrow$ $\omega_{\mathcal{U}}$ = connection matrix of 1-forms over $\mathcal{U}$.

And: $\tilde{\underline{s}} = \underline{s} A \Rightarrow \tilde{\omega} = A^{-1} \omega A + A^{-1} dA$.

On the frame bundle of E, locally, fix $\underline{s}$

$$\begin{array}{ccc} F(E) & \supseteq & \mathcal{U} \times \overset{(x,A)}{\overset{\rotatebox{90}{$\in$}}{}} GL(r) \\ \downarrow & & \downarrow \uparrow B \\ M & \supseteq & \mathcal{U} \end{array} \qquad \begin{array}{c} \hat{\omega}_{\mathcal{U}} = A^{-1} \omega_{\mathcal{U}} A + A^{-1} dA \\ \downarrow B^* \\ \tilde{\omega} \end{array}$$

(Ex: understand this and prove fact below)

Note: $B: \mathcal{U} \to \mathcal{U} \times GL(r)$

$B^*: \Omega^1(\mathcal{U} \times GL(r)) \to \Omega^1(\mathcal{U})$

$\hat{\omega}_{\mathcal{U}} \longmapsto B^* \hat{\omega}_{\mathcal{U}}$

any other local frame $\tilde{\underline{s}} = \underline{s} B$, $B(x) \in GL(r)$

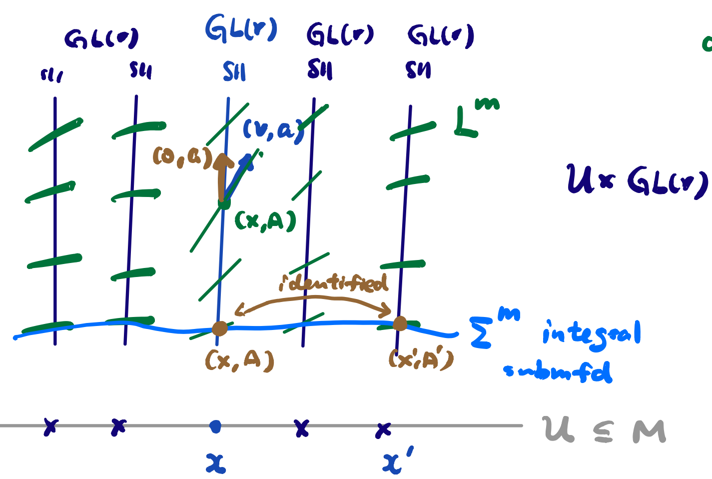
FACT: $\exists$ 1-form (matrix-valued) $\hat{\omega}_{\mathcal{U}}$ on $\mathcal{U} \times GL(r)$ s.t.

$$B^*(\hat{\omega}_{\mathcal{U}}) = \tilde{\omega} = B^{-1} \omega_{\mathcal{U}} B + B^{-1} dB. \quad \left(= \begin{array}{c}\text{conn. matrix} \\ \text{w.r.t. } \tilde{\underline{s}}\end{array}\right)$$

Similarly, one can do it for the curvature 2-forms:

$$\begin{array}{c} \mathcal{U} \times \overset{(x,A)}{\overset{\rotatebox{90}{$\in$}}{}} GL(r) \\ \downarrow \uparrow B \\ \mathcal{U} \end{array} \qquad \begin{array}{l} \hat{\Omega}_{\mathcal{U}} := A^{-1} \Omega_{\mathcal{U}} A \overset{Ex:}{=} d\hat{\omega}_{\mathcal{U}} + \hat{\omega}_{\mathcal{U}} \wedge \hat{\omega}_{\mathcal{U}} \\ \downarrow B^* \\ \tilde{\Omega} = \text{curvature matrix w.r.t. } \tilde{\underline{s}} = \underline{s} B \end{array}$$

Picture: (Kobayashi-Nomizu)



Consider an m-dim'l distribution on $\mathcal{U} \times GL(r)$

$$L^m_{(x,A)} := \{ (v,a) \mid \hat{\omega}_{\mathcal{U}}(v,a) = 0 \}$$

as matrix

Fact: $\hat{\omega}_{\mathcal{U}}(0,a) = a$

$\Downarrow$

L^m is transverse to fibers.

Motto: Having $\hat{\omega}_{\mathcal{U}}$ on $\mathcal{U} \times GL(r)$ $\Leftrightarrow$ having a connection on $\pi^{-1}(\mathcal{U}) \subseteq E$

<u>Prop</u>: (Equivalent characterizations of locally flat connections) TFAE :

(1) D is a flat connection on E over $\mathcal{U}$ (i.e. $\Omega \equiv 0$)

(2) L^m is an integrable distribution

(3) $\exists\, B : \mathcal{U} \to GL(r)$ s.t. $B^*(\hat{\omega}_{\mathcal{U}}) \equiv 0$ on $\mathcal{U}$

(3') $\exists$ parallel local frames $\tilde{s}_1, \dots, \tilde{s}_r$ on $\mathcal{U}$ (i.e. $D\tilde{s}_\alpha \equiv 0$ on $\mathcal{U}$)